

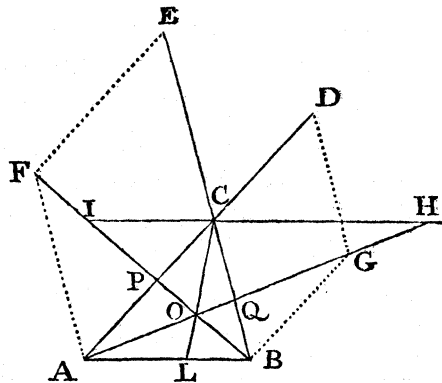
III. *Propositions selected from a Paper on the Division of Right Lines, Surfaces, and Solids. By James Glenie, A. M. of the University of Edinburgh. Communicated by the Astronomer Royal.*

R. June, 1775.

PROPOSITION I. THEOREM.

If from the angles at the base of any right-lined triangle, right lines be drawn to the alternate angles of rhombi, described upon the opposite sides, and applied reciprocally to the sides produced; and from the vertex, through the intersection of these lines, a right line be drawn to meet the base: the segments of the base, made thereby, will have to each other the duplicate proportion of the sides.

LET ACB be any right lined triangle. Let AFEC, CDGB be rhombi, on any two sides AC, CB of this triangle, applied respectively to CB, AC, produced: from the alternate angles EFA, DGB, of which let FA, GA, be right lines drawn to the angles at the base, or third side, AB. Then, if through the intersection O of



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tion of AC to CB (11. E. 5.). But, since the triangles COH, COI, are respectively equiangular to the triangles AOL, LOB, the proportion of CH to IC is equal to the proportion of AL to LB (4. E. 6. and 16. E. 5.). Therefore the proportion of AL to LB is equal to the duplicate proportion of AC to CB (11. E. 5.). 2. E. D.

COR. I. If the triangle be isosceles, the right line drawn from the vertex to the base is perpendicular thereto, and the segments of the base are equal to each other.

COR. II. When the triangle is right-angled, the line drawn from the vertex to the base is always perpendicular to it (as appears from 8. E. 6. and its cor.); and the *rhombi* become squares on the sides comprehending the right angle.

COR. III. The segments of the sides adjacent to the base, are respectively third proportionals to the sum of the sides, and the sides themselves.

COR. IV. The segments of the sides adjacent to the vertex are equal to each other, and each of them is a fourth proportional to the sum of the sides, and the sides themselves(*).

COR.

(a) And it may be added, a mean in proportion between the two segments adjacent to the base. For if a right line AB be any how divided in c, and from the two segments CA, CB, third proportionals to the whole line and each segment respectively, CD, CE, be taken away, the remainders AD, EB, are equal, and each is a mean in proportion between the two CD, CE. For because $AB : AC :: AC : CD$;

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therefore,

COR. V. The segments of the base are proportional to the segments of the sides, which are adjacent to them.

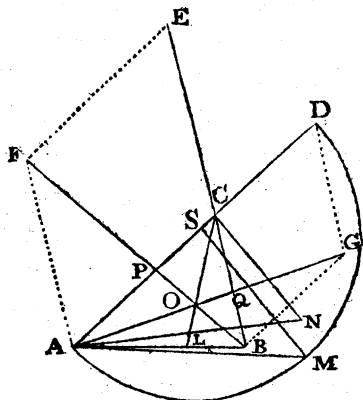
PROPOSITION II.

Let there be any two right lines given. There is an angle which may be made by these lines; such that if, from their extremities which do not meet, right lines be drawn to the alternate angles of rhombi described on them, and reciprocally applied to them when produced; and from the said angle through the intersection of these lines, a right line be drawn to meet the right line joining the said extremities; the segments of this line made thereby, shall be respectively equal to the adjacent segments of the given lines.

therefore, by conversion, $AB : BC = CA : AD$. Again, because $AB : BC = BC : CE$, by conversion $AB : AC = BC : BE$: and by permutation $AB : BC = AC : BE$. Therefore $AC : BE = AC : AD$. Therefore AD and BE are equal. I say, moreover, that each of the two equal lines AD , BE , is a mean in proportion between the two CD , CE . For because $BA : AC = AC : CD$, by division $BC : CA = AD : DC$. Again, because $BA : BC = BC : CE$, converting and dividing $BC : CA = CE : EB$. Therefore, $CE : EB = AD : DC$. Q. E. D. S. HORSLEY.

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LET AC, CB, be any two given right lines, and let CD in AC produced be equal to CB. On AD describe a semi-circle; draw CN at right angles to AD, and equal to CD (I I. E. 1.); join A, N, and apply a right line AM in the semi-circle equal to AN (I. E. 4.). From the point M draw the right line MS at

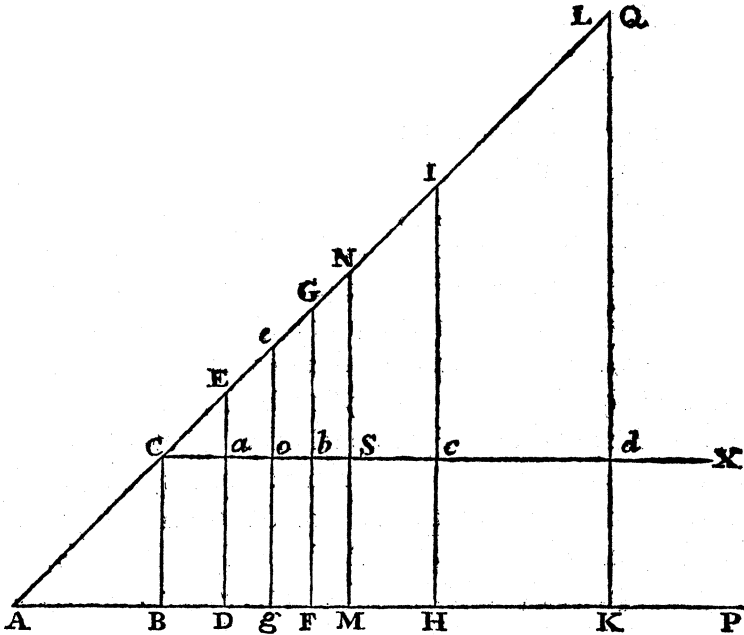


right angles to AD (1. E. 4.) Make a triangle ACB, having its sides equal to AC, AS, and CB (by 22. E. 1.); and ACB is the angle required to be found; and the segments AL, LB, of the right line AB joining the extremities A and B of the given lines are respectively equal to the segments AP, BQ, of the given lines, which are adjacent to them. For the square on BC hath to the square on AC the duplicate proportion of BC to AC (cor. 1. to 20. E. 6.); that is, the proportion of BL to LA (prop. 1.). Wherefore the squares on AC, CB; that is, the square on AN or AM (47. E. 1.) hath to the square on AC the proportion of AB to AL (18. E. 5.). But the square on AN or AM is equal to the rectangle contained by AD, AS (8. and 17. E. 6.). Wherefore the rectangle contained by AD, AS, hath to the square on AC, the proportion of AB to AL; that is, the proportion of the rectangle contained by AB, AS to the rectangle contained by AL, AS (1. E. 6.). Consequently, the proportion of the rectangle AD, AS, to the rectangle

rectangle AB, AS; that is, the proportion of AD to AB (1. E. 6.) is equal to the proportion of the square on AC to the rectangle AL, AS (16. E. 5.) But AD hath to AC the proportion of AC to AP (cor. 3. to prop. 1.). Therefore, the rectangle AD, AC, hath to the square on AC the proportion of the rectangle AC, AS, to the rectangle AP, AS (1. E. 6.). Therefore AD hath to AS or AB the proportion of the square on AC to the rectangle AP, AS (16. E. 5. and 1. E. 6.). Hence the rectangle AL, AS, is equal to the rectangle AP, AS (11. and 16. E. 5.), and AL, AP, are consequently equal. But as AL to AP, so is BL to BQ (by cor. 5. prop. 1.). Therefore BL, BQ, are likewise equal. *Q. E. D.*

PROPOSITION III. PROBLEM I.

To multiply the square of a given finite right line by any number.



UPON an indefinite right line AP set off the given right line AB; draw BC at right angles to AP and equal to AB; and from A through c draw an indefinite right line AQ. Take AD equal to AC, and draw DE parallel to BC; AF equal to AE; and draw FG parallel to BC, and so on. Then it appears (from 47. E. I.), that the square of AC is

is equal to the square of AB multiplied by 2; the square of AE equal to the square of AD or AC multiplied by 2; that is, equal to the square of AB multiplied by 4, and so on. Thus the squares of $AC, AE, AG, AI, AL, \&c.$ are respectively equal to the square of AB multiplied by the terms of the following series 2, 4, 8, 16, 32, 64, $\&c.$ where the sixty-third term gives the square of AB multiplied by the last term of *SESSA'S* Series for the Chess-board.

If CX be drawn parallel to AP , the squares of $Aa, Ab, Ac, Ad, \&c.$ will be respectively equal to the square of AB multiplied by 3, 5, 9, 17, 33, 65, 129, $\&c.$ Also if Ag be taken equal to Aa , and ge be drawn parallel to BC , and this be repeated, the squares of $Ae, \&c.$ will be equal respectively to the square of AB multiplied by 6, 12, 24, 48, $\&c.$ And the squares on $Ao, \&c.$ will be equal to square on AB multiplied by 4, 7, 13, 25, 49, $\&c.$ In like manner, if AM be taken equal to Ab , and MN be drawn parallel to BC , the squares on $AN, \&c.$ will be equal respectively to the square on AB multiplied by 10, 20, 40, 80, 160, $\&c.$ And the squares on $As, \&c.$ will be equal respectively to the square on AB multiplied by the terms of the following series: 6, 11, 21, 41, 81, 161, $\&c.$

In the same way, if right lines be drawn from $E, e, G, N, I, L, \&c.$ there will arise numberless other series. And if BC be taken equal to AB multiplied by any number, furd, fractional, or mixed, there will be obtained a great variety of series, consisting respectively of terms, which
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are furd, fractional, or mixed. And by dividing BC, DE, *ge*, FG, MN, HI, KL, in different ways, according to pleasure, we may apply the same method to fractional numbers, without altering the magnitude of BC. Thus, if BC be bisected, and a right line be drawn through the point of bisection parallel to AP, there will be found right lines, the squares on which are respectively equal to the square on AB multiplied by a great number of fractions, having four for their common denominator, and so on.

PROPOSITION IV. PROBLEM II.

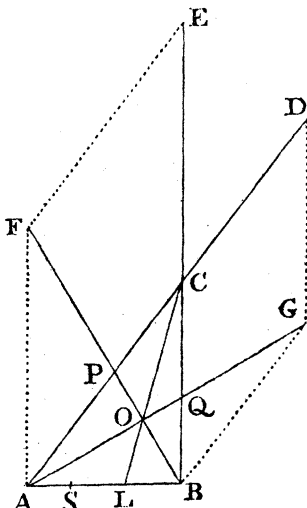
To find a right line, the square on which shall be equal to the square on a given right line, divided by any number.

IF, using the figure of the immediately preceding problem, we suppose the given right line to be denoted by AL, the squares on AK, AH, AF, AD, AB, &c. will respectively be equal to the square on AL multiplied by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1024}$, &c. or divided by 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, &c.; and so on for other numbers, whole, furd, fractional, or mixed.

PROPOSITION V. PROBLEM III.

To cut off from a given right line a part expressed by any odd number.

LET AB be the given right line. At right angles to it, at one of its extremities B, draw an indefinite right line BE. Let n be the given odd number, which expresses the part of AB which is to be cut off. Take BC such a right line (prob. 8.), that the square on it shall be equal to the square on AB, multiplied by the number $\frac{n-1}{2}$. Draw CL as in the first theorem, and take LS equal to LB. I say, AS is that part of AB, which is expressed by the odd number n .

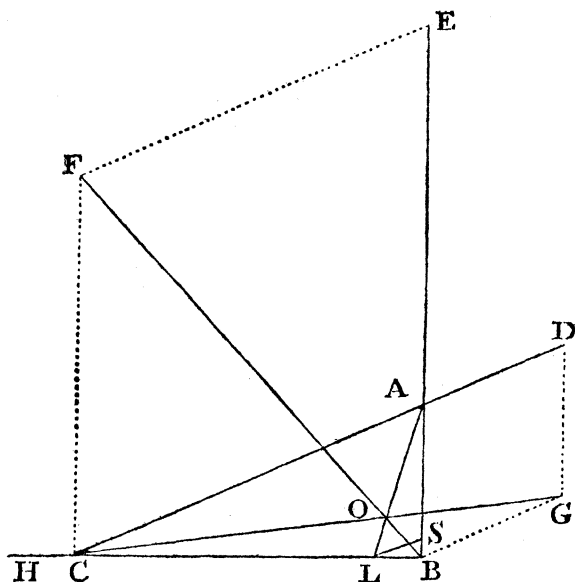


For the square on AC being equal to the squares on AB, BC, is equal to the square on AB multiplied by the number $\frac{n-1}{2} + 1$, or $\frac{n+1}{2}$. Therefore it appears (from prop. 1. and cor. 1. to 20. E. 6.), that AL is to LB as $\frac{n-1}{2} + 1$ to $\frac{n-1}{2}$. Consequently, AS is equal to the part required. Q. E. F.

Thus, if the square on BC be supposed successively equal to the square on AB multiplied by the terms of the series 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, &c. the numbers of the several parts denoted by AS, will be 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, &c. which series comprehends all odd numbers after 9, and might have begun from 3 had the other series begun from 1.

PROPOSITION VI. PROBLEM IV.

To cut off from a given right line a part expressed by any even number.



Let m denote any even number in general. Draw any indefinite right line BH , and at right angles to it another BE . On BE set off the given right line BA , and from A , with the distance equal to a right line, the square on which is equal to the square on AB multiplied by the number

number $m-1$, intersect BH in some point c. From the vertex A of the triangle BAC draw AL as was directed in prop. 1. and draw LS parallel to CA. I say BL is such a part of BC as is expressed by the number m ; and that BS is the same part of AB. For it appears (from prop. 1.) that BL is to CL as 1 to $m-1$. Wherefore BL is the m^{th} part of BC. And BS is the same part of AB that BL is of BC (4. E. 6.)

Thus if the square on AC be successively denoted by the square on AB multiplied by 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, &c. BS will be successively such a part of AB as is expressed by 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, &c.

PROPOSITION VII. THEOREM II.

If from the angles at the base of any right lined triangle, right lines be drawn to the alternate angles of rhombi described on the other two sides, and reciprocally applied to them produced, and through the intersection of these lines, a right line be drawn from the vertex to the base; the rectangle contained by the sines of the angles at the extremities of one of the sides, will be equal to the rectangle contained by the sines of the angles at the extremities of the other^(b); and the parallelepiped contained by the sines of the

(b) The author means, that $\sin. ACL \times \sin. CAL = \sin. BCL \times \sin. CBL$ (see fig. prop. 1.). For $\sin. ACL : \sin. L = AL : AC$, and $\sin. L : \sin. BCL = BC : BL$. Take N, a third in proportion to AL, AC. Then, because $AC^2 : BC^2 = AL : LB$, N will

the angles of one of those triangles, into which the original one is divided by the said line drawn from the vertex. will be equal to the parallelepiped contained by the sines of the angles of the other.

COR. The two triangles, adjacent to the segments of the base, have to each other the proportion of the two adjacent to the sides containing the vertical angle, or the proportion of the two into which the original triangle is divided; and any one of these pairs of triangles are as similar figures described on the sides, being as the segments of the base, which have to each other the duplicate proportion of the sides.

PROPOSITION VIII. THEOREM III.

If from the angles at the hypotenuse of any right angled right lined triangle, right lines be drawn to the alternate angles of squares described on the sides containing the right angle, and from the point where the right line drawn from the right angle, through their intersection, meets the hypotenuse, right lines be drawn to the points, where these lines meet the sides; the lines so drawn will

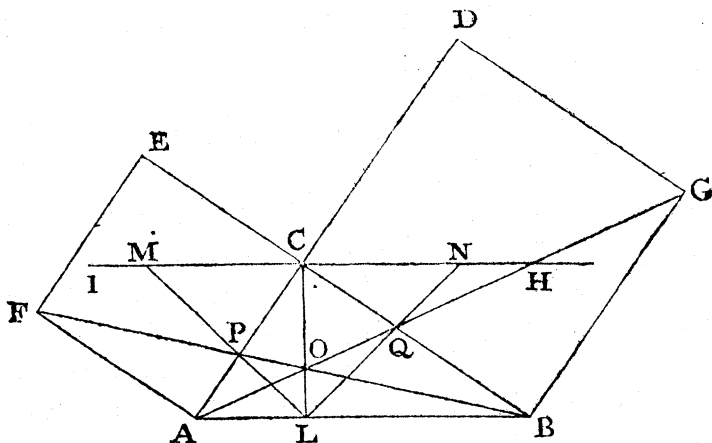
likewise be a third in proportion to LB, BC. Hence, $\sin. ACL : \sin. L = AC : N$, and $\sin. L : \sin. BCL = N : BC$. *Ex æquo perturbate* $\sin. ACL : \sin. BCL = AC : BC$. But $\sin. CBL : \sin. CAL = AC : BC$. Therefore, $\sin. ACL : \sin. BCL = CBL : \sin. CAL$, and $\sin. ACL \times \sin. CAL = \sin. BCL \times \sin. CBL$. Q. E. D. And hence, $\sin. L \times \sin. ACL \times \sin. CAL = \sin. L \times \sin. BCL \times \sin. CBL$, which is the second branch of the proposition. S. HORSLEY.

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make equal angles with the hypotenuse, and the right line drawn from the right angle to meet it; and will likewise have to each other the proportion of the sides containing the right angle.

COR.

(c) For in the triangle ACB, having the angle at c right, construct the figure as in the first proposition, and join LP, LQ. The author affirms, that the angles



$\angle PLA$, $\angle QLB$, and also the angles $\angle CLP$, $\angle CLQ$, are equal, and that $PL : LQ = CA : CB$. Produce LP , LQ , till they meet the right line IH in M , N . Now, because MN , AB , are parallel, therefore $AL : MC = AP : PC$, and $CN : LB = CQ$ (or PC) : QB . Therefore, $AL \times CN : MC \times LB = AP : QB$. But $AP : QB = AL : LB$ (by theor. 1. cor. 5.). Therefore, $AL \times CN : MC \times LB = AL : LB$. Therefore, CN and CM are equal. But because the angle $\angle ACB$ is right, CL is perpendicular to AB (by theor. 1. cor. 2.), and consequently to MN . Therefore the angles $\angle MCL$, $\angle NCL$, are right, and MC being equal to CN , and CL common to the two triangles $\triangle MCL$, $\triangle NCL$, ML will be equal to LN , and the angles $\angle MLC$, $\angle CML$, equal to $\angle NLC$, $\angle CNL$, respectively. Hence it is evident, that the angles $\angle CLP$, $\angle CLQ$, are equal, and each is half a right angle; and likewise $\angle PLA$, $\angle QLB$, are equal, and each half a right angle.

Further,

COR. I. The alternate triangles of those four, which have their vertices in the point, where the right line drawn from the right angle meets the hypotenuse, are similar, and have to each other the proportion of the segments of the hypotenuse, or the duplicate proportion of the sides containing the right angle.

COR. II. Either pair of the adjacent triangles lying on different sides of the right line drawn from the right angle, and having their vertices in the intersection of the right lines drawn from the angles at the hypotenuse, have to each other the proportion of the alternate triangles, having their vertices in the intersection of the first-mentioned line and the hypotenuse.

COR. III. The trapezium or quadrilateral figure formed by the segments of the sides adjacent to the right angle, and the right lines joining their extremities with the intersection of the hypotenuse, and the right line drawn from the right angle to meet it, is capable of being inscribed in a circle; and is divided at the intersection of right lines drawn from the angles at the hypotenuse to the alternate angles of squares, described on the sides containing the right angle, into triangles which are proportional to one another, and when taken two by two, as

Further, the angles, $\angle PLA$, $\angle CQ$, are equal, each being half a right angle. And the angles $\angle PAL$, $\angle LCQ$, are equal (by 8. Elem. 6.). Therefore the angles $\angle APL$, $\angle CQL$, will be equal, and the triangles $\triangle APL$, $\triangle CQL$, similar, and the sides subtending the equal angles proportional. Therefore, $PL : LA = QL : LC$. By permutation, $LP : LQ = LA : LC$. But $LA : LC = AC : CB$ (by 8. Elem. 6.). Therefore, $LP : LQ = AC : CB$. *Q. E. D.* S. HORSLEY.

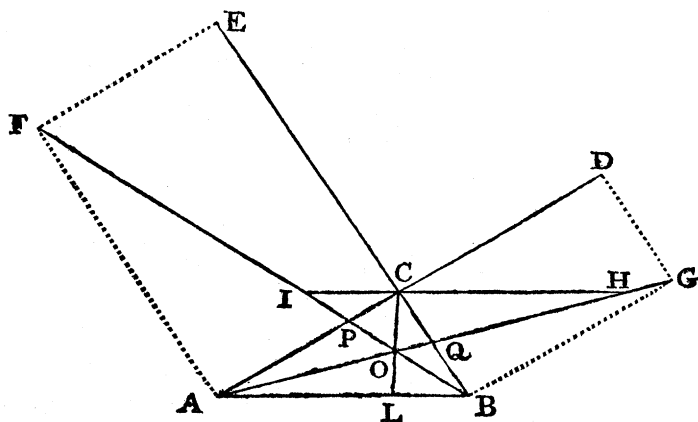
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they lie adjacent on different sides of the diagonal, are proportional to the unequal sides of the trapezium, and to the two triangles into which the diagonal divides it. For $OPL : OPC = LO : OC = OQL : OQC$. Therefore, $OPL : OQL = OPC : OQC = LPC : LQC = LP : LQ$.

PROPOSITION IX. THEOREM IV.

If from the angles at the base of any right lined triangle, right lines be drawn to the alternate angles of rhomboids described on the other two sides, and reciprocally applied to them produced, a right line drawn from the vertex through the intersection of these lines will cut the base into two parts, having to each other the proportion compounded of the proportion of the sides, and of the proportion of the other two lines comprehending the rhomboids.

I SHALL subjoin a demonstration of this theorem, since the first proposition in these papers is only a particular case of it.



Let the triangle be ACB , the base AB , the rhomboids $ACEF$, $CCDG$; and let the right lines BF , AG , be drawn. Then, if from the vertex c through their intersection O , a right line COL be drawn to meet the base, the segments AL , LB , thereof will have to each other the proportion compounded of the proportions of AC to CB , and of CE to CD . For through the vertex c let a right line ICH be drawn parallel to AB , to meet BF , AG , produced, if necessary. Then, since the triangles CQH , CPI , are respectively equiangular to the triangles AQB , APB (15. and 29. E. 1.); the proportions of CH to AB , and of AB to IC are respectively equal to the proportions of CQ to QB , and of AP to PC (4. E. 6.). But the proportion of CH to IC is compounded of the proportions of CH to AB and of AB to IC ; and consequently is equal to a proportion compounded of the proportions of CQ to QB and of AP to PC . And since the triangles ACQ , APF , are respectively equiangular to

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the triangles BQG, BPC (15. and 29. E. 1.); the proportions of CQ to QB, and of AP to PC, are respectively equal to the proportions of AC to BG or CD, and of AE or CE to CB. Wherefore the proportion of CH to IC is equal to the proportion compounded of the proportions of AC to CD and of CE to CB, or of AC to CB and of CE to CD. But since the triangles COH, COI, are respectively equiangular to the triangles AOL, LOB, the proportion of CH to IC is equal to the proportion of AL to LB (4. E. 6.). Therefore the proportion of AL to LB is equal to the proportion compounded of the proportions of AC to CB and of CE to CD. 2. E. D.

SCHOLIUM. If CE, CD, be equal to each other, AL hath to LB the proportion of AC to CB, and CL bisects the angle ACB; if CE have to CD the inverse proportion of AC to CB, AL is equal to LB; if CE have to CD the proportion of AC to CB, AL hath to LB the duplicate proportion of AC to CB; and universally, if CE have to CD any multiplycate proportion, n , of AC to CB, AL hath to LB such a multiplycate proportion of AC to CB as is expressed by the number $n+1$. And if CE have to CD any multiplycate proportion m of CB to AC, AL will have to LB such a multiplycate proportion of CB to AC, as is expressed by the number $m-1$.